Linking number from a topologically massive p-form theory

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Abstract

We show that the linking number of two homologically trivial disjoint p and $(D - p - 1)$-dimensional submanifolds of a D-dimensional manifold can be derived from the topologically massive BC theory in low energy regime.

Antisymmetric tensor fields arise in string theory [1] and supergravity [2] and play an important role in dualization [3, 4, 5]. They can be viewed as the components of a p-form field B given by

$$B = \frac{1}{p!} B_{\mu_1 \dots \mu_p} dx^{\mu_1} \dots dx^{\mu_p}. \tag{1}$$

The theory involving a p-form field B and a $(D - p - 1)$-form field C was first introduced by Horowitz [6] and Blau and Thompson [7]. Horowitz's

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theory does not involve any local dynamics. He was in fact interested in generalizing Witten's idea [8] – who proved the equivalence between the three dimensional Einstein action and the non-abelian Chern-Simons term – to an arbitrary dimension. Horowitz treated a class of models that are invariant under diffeomorphism, and that naturally bring "three dimensional gravity included as a special case". In [9], Horowitz and Srednicki used the same model to provide a definition of generalized linking number of p-dimensional and $(D - p - 1)$-dimensional surfaces in a D-dimensional manifold. Later, making use of variational method, Oda and Yahikozawa [10] obtained the same result and generalized it to the nonabelian case.

The introduction of dynamical terms for a p-form field B and a $(D - p - 1)$-form field C leads to topologically massive theories for abelian [11] and non-abelian [12] gauge theories. These theories are a generalization of the topological mass generation mechanism in three dimensions proposed by Deser, Jackiw and Templeton with the Chern-Simons term [13]. This also generalizes the abelian topological mass mechanism in $D = 4$ constructed with a 2-form and a vector field with a BF term [14]. We emphasize here that the non-abelian construction proposed in [12] does not describes a topolocally massive BF model in D dimensions. The authors did not include the Yang-Mills term, since they consider a flat connection. The non-abelian topological massive Yang-Mills theory with no flat connection was constructed in [15] and [16], in four and D dimensions, respectively.

In this paper we analyze the local effects in the correlation function $\langle B(x)C(y)\rangle$ of the topologically massive abelian BC model integrated over two homologically trivial disjoint submanifolds. We show that the linking number can be derived from the topologically massive BC theory, in the low energy regime generalizing in part the results in [9] and extending to D dimensions the 3-dimensional case [17].

We follow closely the notation and conventions adopted in [18]. We use the form representation for fields with the usual Hodge $*$ operator, which maps a p-form into a $(D - p)$-form and $** = (-1)^{p(D-p)+1}$. The adjoint operator acting in a $p-$form is defined as $d^\dagger = (-1)^{Dp+D} * d*$ [19], where $d = dx^\mu (\partial/\partial x^\mu)$ is the exterior derivative and D is the dimension of a flat manifold $\mathcal{M}_D$ without boundary with metric $g_{\mu\nu} = \text{diag}(- + + \cdots + ++)$. The inner product of two p-forms fields A and B are defined by

$$(A, B) = \int A(x) \wedge *B(x) = \int_M \frac{1}{p!} A(x)_{\mu_1..\mu_p} B(x)^{\mu_1..\mu_p} d^D x. \qquad (2)$$

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The $*d$ operator maps a p-form into a $(D-p-1)$-form and has the properties

$$(\Omega_p, *d\Omega_{D-p-1}) = (-1)^{Dp+1}(\Omega_{D-p-1}, *d\Omega_p), \tag{3}$$

$$(\Omega_p, *d*d\omega_p) = (\omega_p, *d*d\Omega_p), \tag{4}$$

for any p and $(D-p-1)$-form. We use from now on the rules to forms functional calculus developed in [20]:

$$\frac{\delta A(x)}{\delta A(y)} = \delta_p^D(x-y), \tag{5}$$

with $\delta_p^D(x-y)$ is defined in terms of usual Dirac delta function:

$$\delta_p^D(x-y) = \frac{1}{p!}\delta^D(x-y)g_{\mu_1\nu_1}..g_{\mu_p\nu_p}dx^{\mu_1}\wedge..\wedge dx^{\mu_p}\otimes dy^{\nu_1}\wedge..\wedge dy^{\nu_p}. \tag{6}$$

The linking number between two disjoint submanifolds of $\mathcal{M}_D$ can be defined as

$$L(U,V) = \int_U\int_W *\delta_p^D(x-y), \tag{7}$$

where U and V are boundaries of submanifolds Z and W, namely, $U = \partial Z$ and $V = \partial W$. In this expression, x and y are points of U and W respectively, and the $*$ operator acts on the part of $\delta_p^D(x-y)$ defined on W.

We start with the following classical abelian action [12],

$$S = \int_{\mathcal{M}_D}\left(\frac{1}{2}(-1)^r H_B\wedge *H_B + \frac{1}{2}(-1)^s H_C\wedge *H_C + mB\wedge dC\right), \tag{8}$$

where $r = Dp+p+D$, $s = Dp+p+1$, B is a p-form field, C is a $(D-p-1)$-form field both with canonical dimension $(D-2)/2$ and H_B, H_C are their respective field strengths

$$H_B = dB, \tag{9}$$

$$H_C = dC, \tag{10}$$

all them real-valued and m is a mass parameter. The factor (-1) in front of the kinetic terms is required in order to have a positive kinetic energy in the Hamiltonian. As claimed in [12], the model just describe a topologically

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massive BC model denoted by $TMBC$. Note that for $D = 4$ and $p = 1$ we recover the topologically massive BF model [14]. The action is clearly invariant under the gauge transformations

$$\delta B = d\Omega, \tag{11}$$
$$\delta C = d\Theta, \tag{12}$$

where Ω and Θ are $(p-1)$-form and $(D-p-2)$-form gauge parameters. These gauge transformations are reducible, $i.e.,\Omega'$ and Θ' given by

$$\Omega' = \Omega + d\omega, \tag{13}$$
$$\Theta' = \Theta + d\theta, \tag{14}$$

are also honest gauge parameters satisfying (11) and (12) respectively, since $d^2 = 0$. Naturally, the same holds to ω, θ, etc. So, in order to construct the action to be quantized, one has to introduce ghosts and ghosts for ghosts and so on.

Let us write the action in a more compact form. We introduce a doublet $\Phi(x)$, with $B(x)$ and $C(x)$ being the components fields:

$$\Phi(x) = \begin{pmatrix} \Phi_1(x) \\ \Phi_2(x) \end{pmatrix} = \begin{pmatrix} B(x) \\ C(x) \end{pmatrix}. \tag{15}$$

The inner product between two doublets is defined by

$$(\Phi, \Psi) = (\Phi_1, \Psi_1) + (\Phi_2, \Psi_2) = (\Psi, \Phi). \tag{16}$$

Then, making the use of Eqs. (3) and (4), we have

$$S_{TMBC} = \frac{m}{2} \left(\Phi, *^{-1} d\mathcal{O}\Phi \right), \tag{17}$$

where

$$\mathcal{O} = \begin{pmatrix} (-1)^{Dp+D+1} * d/m & 1 \\ (-1)^{Dp+1} & (-1)^{Dp+D+1} * d/m \end{pmatrix}. \tag{18}$$

We are interested in the computation of

$$\left\langle \int_U B(x) \int_V C(y) \right\rangle_{TMBC}. \tag{19}$$

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In order to obtain this correlation function, we must deal with the gauge-fixed action.

The gauge fixed action becomes,

$$S_{gf} = \frac{m}{2} \left(\Phi, *^{-1} d\mathcal{O}\Phi \right) + (L, d * \Phi) + \dots, \tag{20}$$

where the dublet

$$L(x) = \begin{pmatrix} L_1(x) \\ L_2(x) \end{pmatrix}, \tag{21}$$

is the Nakanishi-Lautrup field introduced to implement the evaluation of path integral. Note that L_1 and L_2 are a $(D-p+1)$-form and a $(p-2)$-form respectively. The functional is written as

$$Z = \int \mathcal{D}X e^{iS_{gf}}, \tag{22}$$

where $\mathcal{D}X = \mathcal{D}B\mathcal{D}C\mathcal{D}L_1\mathcal{D}L_2 \cdots$ is the functional measure. From the functional identities

$$\frac{1}{Z} \int \mathcal{D}X \frac{\delta}{\delta\Phi(y)} \left[\Phi(x) e^{iS_{gf}} \right] = 0, \tag{23}$$

and

$$\frac{1}{Z} \int \mathcal{D}X \frac{\delta}{\delta\Phi(y)} \left[L(x) e^{iS_{gf}} \right] = 0, \tag{24}$$

we have

$$im \left\langle \Phi(x) *^{-1} d\mathcal{O}\Phi(y) \right\rangle \pm i \left\langle \Phi(x) d * L(y) \right\rangle + \delta_{p,D-p-1}^D (x-y) = 0, \tag{25}$$

$$\left\langle L(x) *^{-1} d\mathcal{O}\Phi(y) \right\rangle \pm \left\langle L(x) d * L(y) \right\rangle = 0, \tag{26}$$

where

$$\delta_{p,D-p-1}^D (x-y) = \frac{\delta\Phi(x)}{\delta\Phi(y)} = \begin{pmatrix} \delta_p^D(x-y) \\ \delta_{D-p-1}^D(x-y) \end{pmatrix}, \tag{27}$$

and the correlation function of two dublets is taken as being

$$\langle \Phi(x)\Psi(y) \rangle = \begin{pmatrix} \langle \Phi_1(x)\Psi_1(y) \rangle \\ \langle \Phi_2(x)\Psi_2(y) \rangle \end{pmatrix}. \tag{28}$$

To compute these correlation functions, we must invert the operator $\mathcal{O}$. But $\mathcal{O}^{-1}$ has local and non-local terms. To get rid of non-local terms, one has to

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be concerned with low energy regime. So, to get the local terms of $\mathcal{O}^{-1}$, we expand in powers of $*d/m$:

$$\mathcal{O}^{-1} = \begin{pmatrix} (-1)^{D+1} * d/m & (-1)^{Dp+1} \\ 1 & (-1)^{D+1} * d/m \end{pmatrix} \alpha, \tag{29}$$

where

$$\alpha = \Sigma_{n=0}^{\infty} (-1)^{n(Dp+1)} (*d/m)^{2n}. \tag{30}$$

Since L is Nakanishi-Lautrup field, $\langle L(x)L(y)\rangle = 0$. Then, from Eq. (26), we have

$$\langle d\Phi(x)L(y)\rangle = 0, \tag{31}$$

and consequently,

$$\left\langle \int_U B(x) \int_W *d * L_1(y) \right\rangle = 0. \tag{32}$$

Using this identity and Eq. (25) we arrive at

$$m \left\langle \int_U B(x) \int_V C(y) \right\rangle + (-1)^{Dp+D+1} \left\langle \int_U B(x) \int_V *dB(y) \right\rangle = iL(U,V), \tag{33}$$

where we have used the Eq. (7). To evaluate the second term of the equation above, we take $x \neq y$ in the equation (25) and apply $\mathcal{O}^{-1}$ on it:

$$\langle \Phi(x) * d\Phi(y)\rangle = \frac{\pm 1}{m} \left\langle \Phi(x)\mathcal{O}^{-1} d * L(y) \right\rangle. \tag{34}$$

In low energy regime $\mathcal{O}^{-1} d* = \beta d*$, where

$$\beta = \begin{pmatrix} 0 & (-1)^{Dp+1} \\ 1 & 0 \end{pmatrix}. \tag{35}$$

Writing Eq. (34) in components and integrating over U and V, it is clear that

$$\left\langle \int_U B(x) \int_V *dB(y) \right\rangle = \frac{\pm 1}{m} \left\langle \int_U B(x) \int_V d * L_2(y) \right\rangle = 0. \tag{36}$$

So, we finally have that

$$iL(U,V) = m \left\langle \int_U B(x) \int_V C(y) \right\rangle. \tag{37}$$

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We must enforce that this remarkable result was deduced restricting ourselves to low energy regime. Otherwise, non-local terms would appear and could jeopardize our analysis.

Acknowledgments

Dedicatory
"My wife is a great person and I love her" (R. R. Landim).

References

[1] M. Green, J. Schwarz and E. Witten, Superstring Theory, Cambrige University Press, (1987); J. Polchinski, String Theory, Cambrige University Press, (1998)

[2] P. van Nieuwenhuizen, Phys. Reports 68 (1981) 189.

[3] A. Smailagic and E. Spallucci, Phys. Rev. D61 (2000) 067701.

[4] A. Smailagic and E. Spallucci, Phys. Lett. B489 (2000) 435.

[5] R. Menezes, J. R. S. Nascimento, R. F. Ribeiro, C. Wotzasek, Phys. Lett. B564 (2003) 261.

[6] G. T. Horowitz, Commun. Math. Phys. 125 (1989) 417.

[7] M. Blau and G. Thompson, Phys. Lett. B228 (1989) 64.

[8] E. Witten, Nucl. Phys. B311 (1988) 46.

[9] G. T. Horowitz and M. Srednicki, Commun. Math. Phys. 130 (1990) 83.

[10] I. Oda and S. Yahikozawa, Phys. Lett. B238 (1990) 272.

[11] I. Oda and S. Yahikozawa, Progr. Theor. Phys. 83 (1990) 991.

[12] I. Oda and S. Yahikozawa, Phys. Lett. B234 (1990) 69.

[13] S. Deser, R. Jackiw and S. Templeton, Ann. Phys. 140 (1982) 372.

[14] T. J. Allen , M. J. Bowick and A. Lahiri, Mod. Phys. Lett. A6 (1991) 559.

[15] D. S. Hwang and C.-T. Lee, J. Math. Phys. 38 (1997) 30.

[16] R. R. Landim and C. A. S. Almeida, Phys. Lett. B504 (2001) 147.

[17] V. E. R. Lemes, C. A. Linhares, S. P. Sorella, L. C. Q. Vilar, J. Phys. A32 (1999) 2469.

[18] R. R. Landim, Phys. Lett. B542 (2002) 160.

[19] M. Nakahara, *Geometry, Topology and Physics*, IOP, Bristol, 1990.

[20] Marcio A. M. Gomes, R. R. Landim, J. Phys. A-Math. Gen. 38 (2005) 257.